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Non-equilibrium Ising model with competing Glauber dynamics

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Abstract. We consider a kinetic Ising model with ferromagnetic interactions that evolves in time according to two competing Glauber dynamics at different temperatures. The steady states of this non-equilibrium model are studied by using a dynamic pair approximation. When the temperatures are low enough the system orders in a ferromagnetic state, but if one of the temperatures is allowed to be negative the system may have an antiferromagnetic order. We obtain the phase diagram for the case of a square lattice. In this case we calculate the critical exponent ν by using a mean field renormalization group method. The numerical results indicate that the model falls into the same universality class of the equilibrium Ising model. We also show that one particular case of the non-equilibrium model studied here is equivalent to the majority vote model.

1. Introduction

The Ising model with locally competing Glauber dynamics at different temperatures is one of the simplest non-equilibrium spin models that displays a phase transition. Such a model may be interpreted as a system in contact with two heat baths and was first considered by Garrido *et al* [1]. They showed, via mean field approach and Monte Carlo simulation, that the system can be ordered in a non-equilibrium steady state with the same kind of order exhibited in equilibrium. The critical exponent ν for this model was estimated by Marques [2], by using a mean field renormalization group method, suggesting that the model belongs to the same universality class of the equilibrium Ising model.

In this paper we consider the same model but we allow one of the temperatures to become negative. The heat bath at negative temperature is interpreted as being a device that pumps energy into the system. In that way we are considering an Ising system in contact with a heat bath (at a positive temperature) and that receives a continuous flux of energy from the exterior (heat bath at a negative temperature). This source of energy gives to the non-equilibrium system the possibility of having another kind of order distinct from the equilibrium one. We have shown in an earlier study of a similar model that this far-from-equilibrium stationary state is identified with the antiferromagnetic state although the interactions are ferromagnetic ones [3].

The system follows a stochastic process composed of two Glauber processes, each of them simulating one heat bath. The Glauber process describing the colder heat bath

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will occur with probability p and the other one with probability $q = 1 - p$. Although each Glauber process satisfies detailed balance, the composite process does not, with the exception of the one-dimensional case.

The model was solved by using a dynamical pair approximation [3-5]. Given the limitations of the mean field approach in the prediction of critical exponents we use a mean field renormalization group method to calculate the exponent ν. This technique proved rather useful in dealing with equilibrium phase transitions and was first applied to non-equilibrium systems by Marques [2]. The numerical results indicate that the model falls into the universality class of the equilibrium Ising model in agreement with the prediction by Grinstein *et al* [6].

2. The model

Consider a lattice of N Ising spins with ferromagnetic interactions. The state of the system is represented by $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ where $\sigma_i = \pm 1$. The energy of the state σ is

$$E(\sigma) = -J \sum_{(i,j)} \sigma_i \sigma_j \tag{1}$$

where the summation is over nearest-neighbour pairs and $J > 0$. We set $J = 1$.

Let $P(\sigma, t)$ be the probability of state σ at time t. The evolution of $P(\sigma, t)$ is governed by the master equation [7]

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\sigma, t) = \sum_{i=1}^{N} \{P(\sigma^i, t) w_i(\sigma^i) - P(\sigma, t) w_i(\sigma)\} \tag{2}$$

where $w_i(\sigma)$ is the probability of flipping spin i per unit time and the notation $\sigma^i = (\sigma_1, \sigma_2, \ldots, -\sigma_i, \ldots, \sigma_N)$ has been used.

Let us denote by $\langle f(\sigma) \rangle$ the average of a state function $f(\sigma)$, that is,

$$\langle f(\sigma) \rangle = \sum_{\sigma} f(\sigma) P(\sigma, t). \tag{3}$$

From the master equation we get the following equation for the time evolution of $\langle f(\sigma) \rangle$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle f(\sigma) \rangle = \sum_{i=1}^{N} \langle [f(\sigma^i) - f(\sigma)] w_i(\sigma) \rangle. \tag{4}$$

The equations for the magnetization $\langle \sigma_i \rangle$ of spin i and for the correlation $\langle \sigma_j \sigma_k \rangle$ between the nearest-neighbour spins j and k are therefore given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_i \rangle = \langle (-2\sigma_i) w_i(\sigma) \rangle \tag{5}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_j \sigma_k \rangle = \langle (-2\sigma_j \sigma_k)[w_j(\sigma) + w_k(\sigma)] \rangle. \tag{6}$$

In this paper we consider a stochastic dynamics composed of two competing Glauber processes for which $w_i(\sigma)$ is given by

$$w_i(\sigma) = p w_i^A(\sigma) + q w_i^B(\sigma) \tag{7}$$

where $w_i^A(\sigma)$ and $w_i^B(\sigma)$ are the transition rates for Glauber processes at temperatures T_A and T_B, respectively, and are given by [8]

$$w_i^{A,B} = \frac{1}{2}\left[1 - \sigma_i \tanh\left(\beta_{A,B} \sum_{\delta} \sigma_{i+\delta}\right)\right] \tag{8}$$

where the summation is over the nearest neighbours of site i, and $\beta_{A,B} = 1/T_{A,B}$.

For the one-dimensional case it is possible [1] to write the composite process as being one Glauber process

$$w_i(\sigma) = \tfrac{1}{2}[1 - \sigma_i \tanh(\beta(\sigma_{i+1} + \sigma_{i-1}))] \tag{9}$$

at a temperature $T = 1/\beta$ given by

$$\tanh(2\beta) = p \tanh(2\beta_A) + q \tanh(2\beta_B). \tag{10}$$

Therefore, in this case, the system behaves as if it were in contact with just one heat bath at a temperature T. In other words, the system is described, in the stationary state, by the Gibbs distribution $P(\sigma) \propto \exp\{-\beta E(\sigma)\}$.

3. Pair approximation

Let us consider a bipartite lattice of coordination K. We look for solutions such that $\langle\sigma_i\rangle = m_1$ for any spin i of the sublattice 1, $\langle\sigma_j\rangle = m_2$ for any spin j of the sublattice 2 and $\langle\sigma_i\sigma_j\rangle = r$ for any pair of nearest-neighbour spins. If σ_1 and σ_2 are two nearest-neighbour spins belonging to sublattices 1 and 2, respectively, the pair probability $P_{12}(\sigma_1, \sigma_2)$ and the single-spin probabilities $P_1(\sigma_1)$ and $P_2(\sigma_2)$ can be written as

$$P_1(\sigma_1) = \tfrac{1}{2}(1 + m_1\sigma_1) \tag{11}$$

$$P_2(\sigma_2) = \tfrac{1}{2}(1 + m_2\sigma_2) \tag{12}$$

$$P_{12}(\sigma_1, \sigma_2) = \tfrac{1}{4}(1 + m_1\sigma_1 + m_2\sigma_2 + r\sigma_1\sigma_2). \tag{13}$$

To calculate the averages of the right-hand sides of equations (5) and (6) we need the probability of a cluster composed by a central spin and its K nearest neighbours. If the central spin belongs to the sublattice 1 the probability of such a cluster is approximated by [3–5]

$$P_1(\sigma_1) \prod_j \frac{P_{12}(\sigma_1, \sigma_j)}{P_1(\sigma_1)} \tag{14}$$

where the product is over the nearest-neighbour spins of site 1. A similar expression holds for a site belonging to sublattice 2. We point out that for the one-dimensional case the expression (14) is actually exact for the present model under stationary conditions. This can be verified to be the case by comparing the results we obtain, in the stationary state, by using the pair approximation and the results coming from the exact solution of the Ising chain at a temperature T given by equation (10).

By using the pair approximation we obtain closed equations for m_1, m_2 and r. In order to simplify the equations let us introduce the auxiliary quantities $x_1 = P_1(+) = (1+m_1)/2$, $y_1 = P_1(-) = (1-m_1)/2$, $x_2 = P_2(+) = (1+m_2)/2$, $y_2 = P_2(-) = (1-m_2)/2$, $z = P_{12}(++) = (1+m_1+m_2+r)/4$, $v_1 = P_{12}(+-) = (1+m_1-m_2-r)/4$, $v_2 = P_{12}(-+) = (1-m_1+m_2-r)/4$ and $w = P_{12}(--) = (1-m_1-m_2+r)/4$. By using these quantities, the equations for the time evolution of m_1, m_2 and r are

$$\frac{\mathrm{d}m_1}{\mathrm{d}t}=-m_1+\sum_{l=0}^{K}\binom{K}{l}C_l\left\{\frac{z^l v_1^{K-l}}{x_1^{K-1}}+\frac{v_2^l w^{K-l}}{y_1^{K-1}}\right\} \tag{15}$$

$$\frac{\mathrm{d}m_2}{\mathrm{d}t}=-m_2+\sum_{l=0}^{K}\binom{K}{l}C_l\left\{\frac{z^l v_2^{K-l}}{x_2^{K-1}}+\frac{v_1^l w^{K-l}}{y_2^{K-1}}\right\} \tag{16}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t}=-2r+\sum_{l=0}^{K}\binom{K}{l}D_l\left\{\frac{z^l v_1^{K-l}}{x_1^{K-1}}+\frac{v_2^l w^{K-l}}{y_1^{K-1}}+\frac{z^l v_2^{K-l}}{x_2^{K-1}}+\frac{v_1^l w^{K-l}}{y_2^{K-1}}\right\} \tag{17}$$

where

$$C_l=p\tanh[\beta_A(2l-K)]+q\tanh[\beta_B(2l-K)] \tag{18}$$

and

$$D_l=\frac{1}{K}(2l-K)C_l. \tag{19}$$

Notice that $C_{K-l}=-C_l$ and $D_{K-l}=D_l$.

The stationary solutions are obtained when the right-hand sides of equations (15)–(17) vanish. The paramagnetic state corresponds to the trivial solution $m_1=0$, $m_2=0$ and r solution of

$$r=\frac{1}{2^K}\sum_{l=0}^{K}\binom{K}{l}D_l(1+r)^l(1-r)^{K-l}. \tag{20}$$

This solution exists for any value of T_A, T_B and p. However, it becomes unstable in a certain region of the space (T_A, T_B, p). To obtain the line of instability of this solution we do a linear stability analysis. By defining the variables m_F and m_A by $m_F=(m_1+m_2)/2$ and $m_A=(m_1-m_2)/2$ we get, up to linear terms in the infinitesimal deviations δm_F and δm_A, the equations $\mathrm{d}\delta m_F/\mathrm{d}t=\lambda_F\delta m_F$ and $\mathrm{d}\delta m_A/\mathrm{d}t=\lambda_A\delta m_A$ where the eigenvalues λ_F and λ_A are given by

$$\lambda_F=-1+f_K(r) \tag{21}$$

and

$$\lambda_A=-1-f_K(-r) \tag{22}$$

where $f_K(r)$ is defined by

$$f_K(r)=\frac{1}{2^K}\sum_{l=0}^{K}\binom{K}{l}C_l\left\{\frac{2l}{1+r}-(K-1)\right\}(1+r)^l(1-r)^{K-l}. \tag{23}$$

If $\lambda_F<0$ and $\lambda_A<0$, the paramagnetic state is stable. When λ_F becomes positive a ferromagnetic state arises which is characterized by $m_1=m_2\neq 0$. If λ_A becomes positive instead then there appears an antiferromagnetic state characterized by $m_1=-m_2\neq 0$. Therefore $\lambda_F=0$ and equation (20) gives the para-ferromagnetic transition line whereas $\lambda_A=0$ and equation (20) gives the para-antiferromagnetic transition line.

It is straightforward to show that actually the para-ferromagnetic line is given by equation (20) with $r=1/(K-1)$, that is,

$$(K-1)^{K-1}=\frac{1}{2^K}\sum_{l=0}^{K}\binom{K}{l}D_lK^l(K-2)^{K-l} \tag{24}$$

whereas the para-antiferromagnetic line is given by equation (20) but now with $r=-1/(K-1)$, that is,

$$(K-1)^{K-1}=-\frac{1}{2^K}\sum_{l=0}^{K}\binom{K}{l}D_l K^l (K-2)^{K-l}. \tag{25}$$

We point out that when $q=0$, which corresponds to the equilibrium situation, the *solution of equation (24) gives*

$$\tanh(\beta_A)=\frac{1}{K-1} \tag{26}$$

which gives the Bethe–Peierls (pair) approximation for the equilibrium critical temperature. We remark also that along each critical line the energy of the system, which is $N(-Kr/2)$, is a constant since r is a constant, but this result may be an artifact of the mean field approximation.

4. Phase diagram

We apply the results obtained in the previous section to the case of a square lattice ($K=4$). The critical lines are given by

$$p[17\tanh(4\beta_A)+20\tanh(2\beta_A)]+q[17\tanh(4\beta_B)+20\tanh(2\beta_B)]=\pm 27 \tag{27}$$

the upper sign corresponding to the para-ferromagnetic line and the lower to the *para-antiferromagnetic line.* Figures 1–6 show the phase diagram in the variables $T_A=1/\beta_A$ and q for several values of $T_B=1/\beta_B$. For all cases, when $q=0$ (equilibrium case), the system is in the ferromagnetic state if $T_A<T_c$ and in the paramagnetic state if $T_A\geqslant T_c$ where $T_c=2/\ln 2$ is the critical temperature in the Bethe–Peierls (pair) approximation. When $0<T_B<T_c$ (figures 1 and 2), the system is always ordered if $T_A<T_c$ and may be ordered for $T_A>T_c$. When $|T_B|>T_c$ (figures 3–5) the system is always disordered if $T_A>T_c$. If $T_A<T_c$, there is a range in q for which the system is in the paramagnetic state even in the case $T_A=0$.

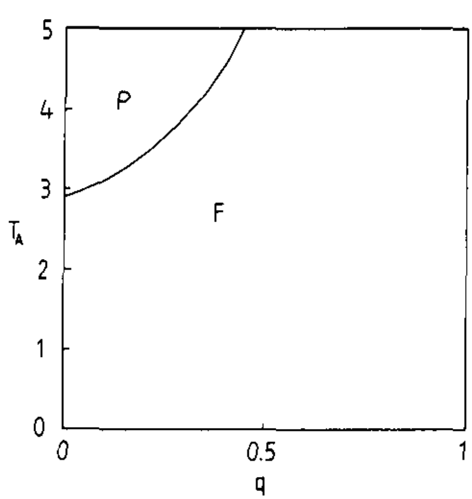

Figure 1. Phase diagram in the variables T_A and q for the case of $T_B=0$. P and F denote the paramagnetic and ferromagnetic phases.

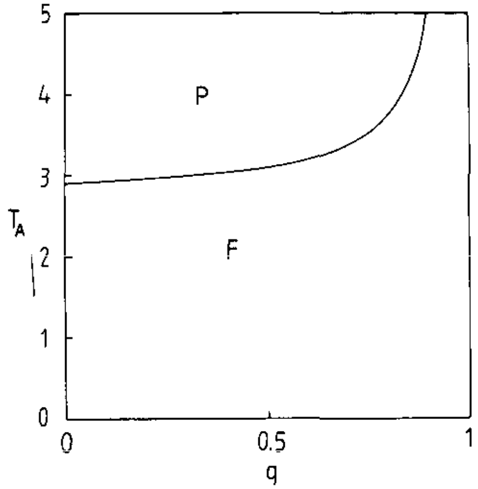

Figure 2. The same as figure 1 for the case of $T_B = 2.7 < T_c$.

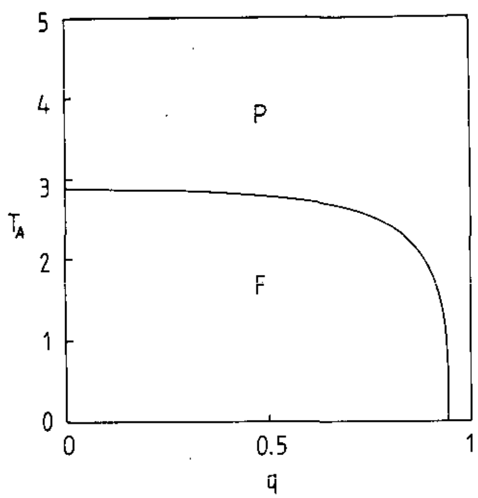

Figure 3. The same as figure 1 for the case of $T_B = 3 > T_c$.

When $-T_c < T_B < 0$, the phase diagram is similar to the extreme case in which $\beta_B \to -\infty$ (figure 6). This corresponds to the case where the system is in contact with a heat bath at a temperature $T_A = 1/\beta_A$ and that receives a continuous flux of energy from the exterior. In this case besides the ferromagnetic and the paramagnetic phases there appears an antiferromagnetic state when the flux of energy is high. The critical lines are given by

$$p[17\tanh(4\beta_A) + 20\tanh(2\beta_A)] - 37q = \pm 27 \tag{28}$$

the upper sign corresponding to the para-ferromagnetic line and the lower one to the para-antiferromagnetic line. The critical lines cross the q-axis at $q = 5/37$ and $q = 32/37$.

The antiferromagnetic state found here should be expected since the spin-flip probability of the Glauber dynamics depends on the ratio between the spin coupling and the bath temperature. When one of the temperatures is negative, the system can

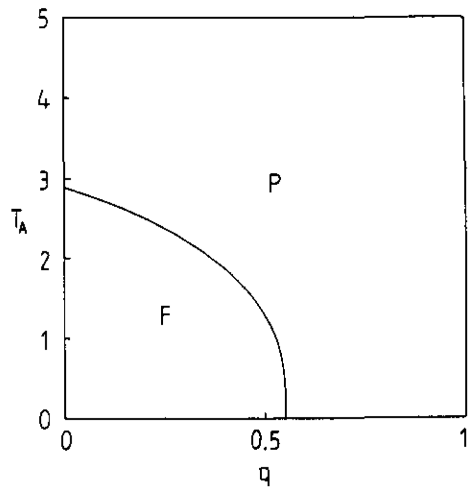

Figure 4. The same as figure 1 for the case of $T_B = 5 > T_c$.

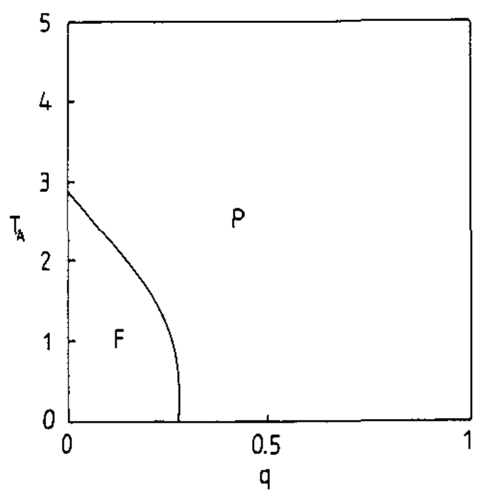

Figure 5. The same as figure 1 for the case of $T_B = \infty$.

be seen as a kinetic Ising model with competition between ferromagnetic and antiferromagnetic Glauber dynamics, both at positive temperatures.

5. Renormalization group calculation

We consider here only the para-ferromagnetic transition line. The results for the para-antiferromagnetic critical line can be obtained by a similar procedure. Following the scheme proposed by Marques [2] we start by considering two separate clusters of spins: cluster I consisting of spin σ_0 and cluster II consisting of spins σ_1 and σ_2. Let us denote by $P_{\mathrm{I}}(\sigma_0)$ and by $P_{\mathrm{II}}(\sigma_1, \sigma_2)$ their respective probabilities. We suppose that each nearest-neighbour spin of cluster I has a probability $(1+b')/2$ of being up and a probability $(1-b')/2$ of being down. The corresponding quantities for the cluster II are $(1+b)/2$ and $(1-b)/2$, respectively.

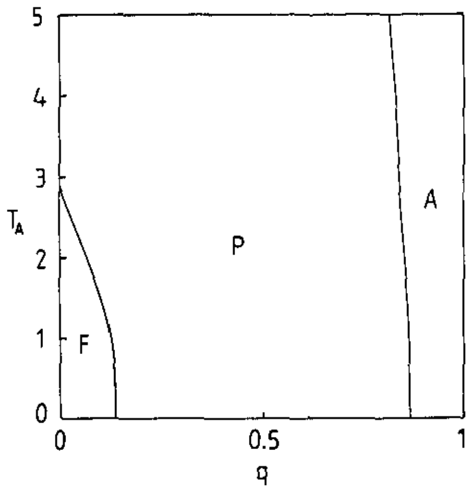

Figure 6. The same as figure 1 for the case of $\beta_B = -\infty$. In this case the system may be ordered in the antiferromagnetic (A) state.

The 'effective fields' b and b' approach zero near the transition line and so do the stationary values of the quantities $P_I(+) - P_I(-)$ and $P_{II}(++) - P_{II}(--)$, which we denote by m' and m, respectively. The MFRG scaling assumption is that $m'/m = b'/b$ near the critical line. This equation is interpreted as a renormalization group recursion relation for the parameters (β'_A and β'_B for cluster I and β_A and β_B for cluster II). For the square lattice one gets up to linear order terms in b' and b

$$m' = \frac{b'}{2}[p(\tanh 4\beta'_A + 2\tanh 2\beta'_A) + q(\tanh 4\beta'_B + 2\tanh 2\beta'_B)] \tag{29}$$

and

$$m = \frac{3b[p(\tanh 4\beta_A + 2\tanh 2\beta_A) + q(\tanh 4\beta_B + 2\tanh 2\beta_B)]}{8 - p(\tanh 4\beta_A + 2\tanh 2\beta_A) - q(\tanh 4\beta_B + 2\tanh 2\beta_B)}. \tag{30}$$

The recursion relation $\beta_A \to \beta'_A$ is obtained by setting $\beta'_B = \beta_B$ and by using the scaling assumption. The corresponding fixed point equations yield the critical lines. The phase diagrams given by this method is very similar to those calculated in the pair approximation. For instance, when $\beta_B \to -\infty$ we get $2p[\tanh(4\beta_A) + 2\tanh(2\beta_A)] - 3q = \pm 2$, instead of equation (28), the intersection with the q-axis occurring at $q = \frac{1}{6}$ and $q = \frac{5}{6}$.

The exponent ν is obtained by $l^{1/\nu} = \partial\beta'_A/\partial\beta_A$ along the critical line. The standard choice for the scaling factor is in this case $l = \sqrt{2}$ since the ratio between the number of spins in cluster II and cluster I is 2. This leads to $\nu \approx 1.2$ for all points on the critical line, the discrepancy from one point to another being less than 0.1%. The same MFRG approximation when applied to the equilibrium Ising model gives also $\nu \approx 1.2$. Although these estimates differ from the exact value $\nu = 1$ for the equilibrium Ising model they are within the deviations that one should expect when dealing with clusters of small sizes. This seems to confirm that the model studied here falls in fact into the universality class of the equilibrium Ising as already found in other cases and in agreement with the argument by Grinstein *et al* [6].

6. Majority vote model

In the limit of $\beta_A \to \infty$ and $\beta_B \to -\infty$ the model studied here is equivalent to the majority vote model [9, 10]. Taking these limits in equation (8), we get

$$w_i(\sigma) = \frac{1}{2}\left[1 - (p-q)\sigma_i S\left(\sum_\delta \sigma_{i+\delta}\right)\right] \tag{31}$$

where the function $S(x)$ is defined by $S(x) = \mathrm{sign}(x)$ if $x \neq 0$ and $S(0) = 0$. Thus, the ith spin flips to the majority sign of its nearest-neighbour spins with probability p and to the minority with probability q. If the number of positive and negative signs are equal, the probability of flipping is $\frac{1}{2}$.

In one dimension, it follows from equations (9) and (10) that the majority vote process is equivalent to a Glauber process at an inverse temperature $\beta = \frac{1}{4}\ln(p/q)$ so that the stationary state will be described by the equilibrium nearest-neighbour Ising model. Therefore, one has a disordered state as long as $0 < q < 1$.

Within the pair approximation the following conclusions regarding the majority vote model may be drawn. In a bipartite lattice, the majority vote model has a ferromagnetic order for $q < q_c$ and an antiferromagnetic order for $q > 1 - q_c$. For $q_c \leq q \leq 1 - q_c$, the system is disordered. From equations (24), (18) and (19) we get

$$q_c = \frac{1}{2}\left\{1 - \frac{2^K (K-1)^{K-1}}{\sum_{l=0}^K \binom{K}{l} |2l - K| K^{l-1} (K-2)^{K-l}}\right\}. \tag{32}$$

For $K = 4$ we get $q_c = \frac{5}{37} \approx 0.135$. For $K = 6, 8, 10$ we obtain $q_c = 0.214$, 0.258, 0.287, respectively. When $K \to \infty$ we have $q_c \to \frac{1}{2}$, the asymptotic behaviour being given by

$$q_c = \tfrac{1}{2}\{1 - (\pi/2K)^{1/2}\}. \tag{33}$$

The majority vote model is a particular case of a general class of polling models introduced by Gray [10]. It is possible to argue [10] that the majority-vote model on a square lattice has two phases at sufficiently small q. Upper bounds for the critical parameter q_c can be obtained for any regular lattice by using the following result [11]. Suppose the spin-flip probability is written in the form

$$w_i(\sigma) = \frac{1}{2}\left[1 - \sigma_i \sum_A c_A \sigma_A\right] \tag{34}$$

where σ_A is the product of spin variables belonging to a cluster of sites A. Then, the system has only one stationary state when $\sum_A |c_A| < 1$. The condition $\sum_A |c_A| = 1$ gives then an upper bound for q_c. Applying this result for the cases $K = 4, 6, 8, 10$ we get the upper bounds $q = 0.25$, 0.357, 0.417, 0.452, respectively. All these values are well above the respective results we have obtained by the pair approximation.

7. Conclusion

We have studied a stochastic Ising model with two competing Glauber dynamics at different temperatures. The steady states were obtained by using a dynamical pair approximation. When both temperatures are positive the system orders in a ferromagnetic state if the temperatures are low enough. When one is negative the system may have an antiferromagnetic order although the interactions are ferromagnetic. All transitions are found to be continuous. By using a mean field renormalization group method we have studied the critical exponent ν. The numerical results indicate that the system is in the same universality class of the equilibrium Ising model.

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